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TWO COMBINATORIAL PROPERTIES OF A CLASS OF SIMPLICIAL POLYTOPES

BY

CARL W. LEE

ABSTRACT

Let $f(\mathcal{P}_s^d)$ be the set of all f-vectors of simplicial d-polytopes. For P a simplicial d-polytope let $\Sigma(P)$ denote the boundary complex of P. We show that for each $f \in f(\mathcal{P}_s^d)$ there is a simplicial d-polytope P with $f(P) = f$ such that the simplicial diameter of $\Sigma(P)$ is no more than $f_0(P) - d + 1$ (one greater than the conjectured Hirsch bound) and that P admits a subdivision into a simplicial d-ball with no new vertices that satisfies the Hirsch property. Further, we demonstrate that the number of bistellar operations required to obtain $\Sigma(P)$ from the boundary of a d-simplex is minimum over the class of all simplicial polytopes with the same f-vector. This polytope P will be the one constructed to prove the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes.

1. Introduction

The *f-vector* of a simplicial (convex) d-polytope P is the d-vector $f(P) = (f_0(P), f_1(P), \dots, f_{d-1}(P))$, where $f_j(P)$ is the number of j-faces (j-dimensional faces) of P, $0 \leqq j \leqq d-1$. The *h-vector* of P is the $(d+1)$-vector $h(P) = (h_0(P), h_1(P), \dots, h_d(P))$ defined by

$$h_i(P) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(P), \qquad 0 \leqq i \leqq d,$$

where we take $f_{-1}(P) = 1$. Knowing $h(P)$ is equivalent to knowing $f(P)$ since

$$f_j(P) = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i(P), \qquad 0 \leqq j \leqq d-1.$$

Let $\mathcal{P}_s^d$ denote the set of all simplicial d-polytopes. The set $h(\mathcal{P}_s^d) = \{h(P) : P \in \mathcal{P}_s^d\}$ is completely characterized by McMullen's conditions [1], [11]. Associated with each $h \in h(\mathcal{P}_s^d)$ there is a natural class $\mathcal{P}_s^d(h)$ of all simplicial

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d-polytopes P with $h(P)=h$. It may be fruitful to study how various properties of simplicial polytopes behave with respect to this partition of $\mathcal{P}_s^d$. For example, the Hirsch conjecture for bounded polyhedra is equivalent to the conjecture that $\operatorname{diam}\Sigma(P)\leqq h_1(P)$ for all simplicial polytopes P, where $\operatorname{diam}\Sigma(P)$ is the simplicial diameter of the boundary complex of P. As another example, the boundary complex of any simplicial d-polytope P can be obtained from the boundary of a d-simplex by a finite sequence of bistellar operations. Pachner has shown that the minimum number $l(P)$ of operations required in such a sequence is at least $h_n(P)-1$, where $n=[d/2]$.

In this paper we show that for each class $\mathcal{P}_s^d(h)$ there is a simplicial d-polytope $P=P(d,h)\in\mathcal{P}_s^d(h)$ such that (1) $\operatorname{diam}\Sigma(P)\leqq h_1(P)+1$, (2) P is the boundary of a simplicial d-ball Δ with $f_0(P)$ vertices satisfying $\operatorname{diam}\Delta\leqq h_1(P)-1$, and (3) $l(P)=h_n(P)-1$. This polytope P will be the polytope constructed in [1] to establish the sufficiency of McMullen's conditions. Using a result about a specific class of vertex decomposable simplicial complexes we will deduce that Δ is vertex decomposable and that $\Sigma(P)\backslash v$ is vertex decomposable for a particular vertex v of P, from which properties (1) and (2) will follow. Property (3) will be a consequence of the structure of Δ and some observations about shellings of simplicial balls.

2. Preliminary definitions and facts

A *simplicial complex* on the finite set $V=V(\Delta)$ is a nonempty collection of subsets of V with the property that $\{v\}\in\Delta$ for all $v\in V$ and that $F\in\Delta$ whenever $F\subseteq G$ for some $G\in\Delta$. For $F\in\Delta$ we say F is a *face* of Δ and the *dimension* of F, $\dim F$, equals j if $\operatorname{card} F=j+1$. Such an F will be called a j-*face* of Δ. The *dimension* of Δ, $\dim\Delta$, is $\max\{\dim F : F\in\Delta\}$. If $\dim\Delta=d-1$ we will refer to Δ as a *simplicial* $(d-1)$-*complex*. In this case faces of Δ of dimension 0, 1, $d-2$ and $d-1$ will be called *vertices*, *edges*, *ridges* and *facets* of Δ, respectively. As with polyhedra we denote the number of j-faces of Δ by $f_j(\Delta)$, $0\leqq j\leqq d-1$, and define $f(\Delta)=(f_0(\Delta),f_1(\Delta),\dots,f_{d-1}(\Delta))$ to be the f-*vector* of Δ. We compute the h-*vector* of Δ, $h(\Delta)=(h_0(\Delta),h_1(\Delta),\dots,h_d(\Delta))$, using the same formula as for simplicial d-polytopes. We call Δ *pure* if each maximal face of Δ (with respect to inclusion) is a facet.

Let Δ be a simplicial complex. For $F\in\Delta$ the *link* of F in Δ is the simplicial complex $\operatorname{lk}_\Delta F=\{G\in\Delta : G\cap F=\varnothing, G\cup F\in\Delta\}$. If $F\neq\varnothing$ the *deletion* of F from Δ is the simplicial complex $\Delta\backslash F=\{G\in\Delta : F\not\subseteq G\}$. Let Δ_1 and Δ_2 be simplicial complexes on disjoint vertex sets V_1 and V_2, respectively. The *join* of

Δ_1 and Δ_2 is the simplicial complex on the set $V_1 \cup V_2$ defined by $\Delta_1 \cdot \Delta_2 = \{F_1 \cup F_2 : F_1 \in \Delta_1, F_2 \in \Delta_2\}$. For a finite set F the power set of F will be denoted $\bar{F}$. For convenience we may sometimes write a set $F = \{v_1, v_2, \dots, v_k\}$ as $v_1 v_2 \cdots v_k$ and $\bar{F}$ as $\overline{v_1 v_2 \cdots v_k}$.

By $|\Delta|$ is meant the underlying topological space of the simplicial complex Δ. If $|\Delta|$ is a topological $(d-1)$-dimensional ball (respectively, sphere), we say Δ is a *simplicial* $(d-1)$-*ball* (respectively, *simplicial* $(d-1)$-*sphere*). If Δ is a simplicial sphere then Δ is pure and every ridge of Δ is contained in exactly two facets of Δ. If Δ is a simplicial d-ball then Δ is pure and every ridge of Δ is contained in at most two facets of Δ. In this case write $\partial\Delta$ for the simplicial $(d-1)$-sphere naturally associated with the boundary of $|\Delta|$. This complex is called the *boundary* of Δ and it is known that $\partial\Delta = \bigcup\{\bar{F} : F$ is a ridge of Δ contained in exactly one facet of $\Delta\}$. Each $F \in \partial\Delta$ will be called a *boundary face* of Δ. If $\Delta = \bar{F}$ for some finite set F of cardinality $k \geqq 1$ then Δ is a simplicial $(k-1)$-complex and will be called a $(k-1)$-*simplex*. In fact Δ is a simplicial ball and $\partial\Delta$ is the set of all proper subsets of F.

We will be primarily interested in simplicial complexes related to convex polyhedra. For basic notions in the theory of polyhedra see [4]. If P is a bounded convex polyhedron then P is called a convex *polytope.* A convex d-polytope for which every facet contains exactly d vertices is called a *simplicial* polytope. A convex d-polyhedron with nonempty vertex set for which every vertex lies in exactly d facets is called a *simple* polyhedron. If P is a simplicial d-polytope, the simplicial $(d-1)$-complex $\Sigma(P)$ associated with P is defined to be $\Sigma(P) = \{F \subset V(P)$: conv F is a face of $P\}$, where $V(P)$ is the set of vertices of P and conv F means the convex hull of F. This simplicial complex is called the *boundary complex* of P. Note the obvious fact that $h(P) = h(\Sigma(P))$. For every simple convex d-polyhedron P there is a simplicial $(d-1)$-complex $\Sigma^*(P)$ that is *dual* to P in the sense that there is an inclusion-reversing bijection between the set of faces of $\Sigma^*(P)$ and the set of nonempty faces of P [9], [2]. In the case that P is a simple polytope then $\Sigma^*(P)$ is just $\Sigma(P^*)$, where P^* is a simplicial d-polytope dual to P.

3. Diameters

The *distance* between two vertices of a convex polyhedron is the length of the shortest edge path between them. The *diameter* of the polyhedron P is the maximum distance between any two vertices of P. The *Hirsch conjecture* [3, p. 168], [4, Ch. 16], [5] states that the diameter of a d-polyhedron with ν facets is at most $\nu - d$. While the conjecture has been shown to be false for unbounded polyhedra of dimension greater than three, it is true for all polyhedra of dimension at most three and for all polytopes for which $\nu - d \leqq 5$. It is still an open question as to whether the Hirsch conjecture is true for all polytopes.

For the purposes of placing upper bounds on the diameters of polyhedra it is sufficient to consider simple polyhedra. This suggests examining simplicial complexes that are dual to simple polyhedra. If Δ is a pure simplicial complex, the *distance* between two facets F and G of Δ, $d(F, G)$, is the length k of the shortest simplicial path $F = F_0, F_1, \ldots, F_k = G$ between F and G, where the F_i are facets of Δ and $F_i \cap F_{i-1}$ is a ridge of Δ for $1 \leqq i \leqq k$. The *diameter* of Δ, $\operatorname{diam} \Delta$, is the maximum distance between any two facets of Δ. We say a simplicial $(d-1)$-complex Δ satisfies the *Hirsch property* if $\operatorname{diam} \Delta \leqq f_0(\Delta) - d = h_1(\Delta)$. Clearly, the diameter of a simplicial complex dual to a simple polyhedron equals the edge path diameter of the polyhedron itself. Thus the Hirsch conjecture for simple d-polytopes is equivalent to the conjecture that $\operatorname{diam} \Sigma(P) \leqq \nu - d = h_1(P)$ for every simplicial d-polytope P with ν vertices.

While it would be nice to prove that $\operatorname{diam} \Sigma(P) \leqq h_1(P)$ for the simplicial polytopes $P = P(d, h)$ constructed in [1], so far we have only been able to bound the diameter of $\Sigma(P)$ by $h_1(P)+1$. What makes this result perhaps more intriguing is that there are no known bounds on $\operatorname{diam} \Sigma^*(Q)$ for arbitrary simple d-polyhedra Q with ν facets that are even polynomial in ν and d. On the other hand no example is known of a bounded or unbounded simple d-polyhedron Q for which $\operatorname{diam} \Sigma^*(Q)$ exceeds, say, $2h_1(\Sigma^*(Q))$. Although we do not yet know whether $\Sigma(P(d, h))$ satisfies the Hirsch property, we have been able to show that $\Sigma(P(d, h))$ is realizable as the boundary complex of a simplicial d-ball Δ with the same vertices as $\Sigma(P(d, h))$ such that Δ satisfies the Hirsch property.

Our primary tool in this section will be that of vertex decomposability, a notion defined by Provan and Billera in their studies of the diameters of simplicial complexes [9]. A simplicial $(d-1)$-complex Δ is said to be *vertex decomposable* if it is pure and either Δ is a $(d-1)$-simplex, or else there is a vertex v of Δ such that $\Delta \setminus v$ is a vertex decomposable simplicial $(d-1)$-complex and $\operatorname{lk}_\Delta v$ is a vertex decomposable simplicial $(d-2)$-complex. In this case v is called a *shedding vertex* of Δ.

THEOREM 3.1 (Provan–Billera). *If Δ is a vertex decomposable simplicial $(d-1)$-complex then Δ satisfies the Hirsch property, i.e.*, $\operatorname{diam} \Delta \leqq h_1(\Delta)$.

PROOF. See [9]. □

We will first construct a particular class of vertex decomposable complexes. Let ν, d, p, q be positive integers such that $\nu \geqq d = pq$. Define $V = \{u_1, u_2, \dots, u_\nu\}$ and $\mathscr{E}_\nu^{p,q}$ to be the collection of all subsets F of V of the form $\{u_{i_1}, \dots, u_{i_1+p-1}\} \cup \{u_{i_2}, \dots, u_{i_2+p-1}\} \cup \cdots \cup \{u_{i_q}, \dots, u_{i_q+p-1}\}$, where $i_1 \geqq 1$, $i_q + p - 1 \leqq \nu$, and $i_k + p - 1 < i_{k+1}$, $1 \leqq k \leqq p - 1$. Thus F consists of q blocks of contiguous vertices of V, each block containing p elements. For such an F put $\delta(F) = q - (r-1)/p$, where $r = \min\{i : u_i \notin F\}$. The integer $\delta(F)$ equals the number of blocks of F that are not in their "leftmost possible position."

We totally order the subsets of V by saying, for $F, G \subseteq V$, that $F < G$ if there is some integer $1 \leqq k \leqq \nu$ for which (1) $u_i \in F$ if and only if $u_i \in G$ for all $k+1 \leqq i \leqq \nu$, and (2) $u_k \in G - F$. A subset $\mathscr{B}$ of $\mathscr{E}_\nu^{p,q}$ will be said to satisfy property (β) if $\mathscr{B}$ is nonempty and $F \in \mathscr{B}$ whenever both $F < G$ and $\delta(F) \leqq \delta(G)$ for some $G \in \mathscr{B}$. Denote by $\Delta(\mathscr{B})$ the simplicial $(d-1)$-complex $\bigcup\{\bar{F} : F \in \mathscr{B}\}$. Our interest in these simplicial complexes is due to the following:

THEOREM 3.2. *Suppose $\mathscr{B} \subseteq \mathscr{E}_\nu^{p,q}$ satisfies (β). Then $\Delta(\mathscr{B})$ is vertex decomposable and hence satisfies the Hirsch property.*

PROOF. Let $r = \max\{i : u_i \in \Delta(\mathscr{B})\}$. If $r = d$ then $\Delta(\mathscr{B})$ is a $(d-1)$-simplex and hence vertex decomposable. So assume that $r \geqq d+1$. We will prove that u_r is a shedding vertex of $\Delta(\mathscr{B})$.

We first claim that $\Delta(\mathscr{B}) \backslash u_r$ is pure $(d-1)$-dimensional. For suppose $u_r \notin G \subseteq \Delta(\mathscr{B})$. Let F be a facet of $\Delta(\mathscr{B})$ containing G. If $u_r \notin F$ all is well. If, on the other hand, $u_r \in F$, let F' be the member of $\mathscr{E}_\nu^{p,q}$ obtained from F by replacing u_r with the u_i of highest index less than r such that $u_i \notin F$. Observe that $F' \in \mathscr{B}$ since $F' < F$ and $\delta(F') \leqq \delta(F)$. Thus F' is a facet of $\Delta(\mathscr{B})$ containing G but not u_r.

Now that we know that $\Delta(\mathscr{B}) \backslash u_r$ is pure $(d-1)$-dimensional we see that $\Delta(\mathscr{B}) \backslash u_r = \Delta(\mathscr{B}_1)$, where $\mathscr{B}_1 = \{F \in \mathscr{B} : u_r \notin F\}$. But it is easy to check that $\mathscr{B}_1$ satisfies (β) and so $\Delta(\mathscr{B}_1)$ is vertex decomposable by induction on r.

We must now examine $\mathrm{lk}_{\Delta(\mathscr{B})}\, u_r$, necessarily a pure simplicial $(d-2)$-complex. If $q = 1$ then $\mathrm{lk}_{\Delta(\mathscr{B})}\, u_r = \overline{\{u_{r-p+1}, \dots, u_{r-1}\}}$ which is a $(d-2)$-simplex and thus vertex decomposable. So assume that $q \geqq 2$. Let $\mathscr{B}_2 \subseteq \mathscr{E}_\nu^{p,q-1}$ be given by $F \in \mathscr{B}_2$ if $F \cup \{u_{r-p+1}, \dots, u_r\} \in \mathscr{B}$. Then $\mathrm{lk}_{\Delta(\mathscr{B})}\, u_r = \Delta(\mathscr{B}_2) \cdot \overline{\{u_{r-p+1}, \dots, u_{r-1}\}}$. Again it is straightforward to demonstrate that $\mathscr{B}_2$ satisfies (β). Therefore $\mathrm{lk}_{\Delta(\mathscr{B})}\, u_r$ is vertex decomposable by induction on q and proposition 2.4 of [9] which states that the join of two vertex decomposable simplicial complexes is again vertex decomposable. $\square$

Now let $h \in h(\mathscr{P}_s^d)$ and take $P = P(d,h) \in \mathscr{P}_s^d(h)$ to be the simplicial d-polytope constructed in [1] to derive the sufficiency of McMullen's conditions. In particular $\Sigma(P)$ is the boundary complex $\partial\Delta$ of a simplicial d-ball Δ with vertex set $V' \cup U$, where $V' = \{v_1, \dots, v_{d-2n+1}\}$, $U = \{u_1, \dots, u_{\nu'}\}$, $n = [d/2]$ and $\nu' = h_1 + 2n - 1$. The facets of Δ are of the form $V' \cup F$, where $F \in \mathscr{B}$ for some fixed $\mathscr{B} \subseteq \mathscr{C}_{\nu'}^{2,n}$ that satisfies (β). Thus $\Delta = \bar{V}' \cdot \Delta(\mathscr{B})$ and we have

Corollary 3.3. *If $P = P(d,h)$ then $\Sigma(P) = \partial\Delta$ for some simplicial d-ball Δ with $f_0(P)$ vertices such that* $\operatorname{diam}\Delta \leqq h_1(P) - 1$, *i.e., Δ satisfies the Hirsch property.*

Proof. Use Theorem 3.2 and proposition 2.4 of [9] to conclude that Δ is vertex decomposable. □

Now take Δ' to be the simplicial $(d-1)$-ball $\overline{\{v_2, \dots, v_{d-2n+1}\}} \cdot \Delta(\mathscr{B})$, again vertex decomposable. It is a routine matter to verify that $\Delta = \bar{v}_1 \cdot \Delta'$ and $(\partial\Delta) \backslash v_1 = \Delta'$. Let F_1, F_2 be facets of $\partial\Delta$. If neither F_1 nor F_2 contain v_1 then $d(F_1, F_2) \leqq \operatorname{diam}\Delta' \leqq f_0(P) - 1 - d = h_1(P) - 1$. Assume $v_1 \in F_1$ but $v_1 \notin F_2$. Let G be the unique facet of $\partial\Delta$ not containing v_1 such that $F_1 \cap G$ is a ridge of $\partial\Delta$. Then

$$d(F_1, F_2) \leqq d(F_1, G) + d(G, F_2) \leqq 1 + \operatorname{diam}\Delta' = h_1(P).$$

A similar argument holds if $v_1 \notin F_1$ but $v_1 \in F_2$. Finally assume $v_1 \in F_1$ and $v_1 \in F_2$. Let G_1, G_2 be the facets of $\partial\Delta$ not containing v_1 such that $F_1 \cap G_1$, $F_2 \cap G_2$ are ridges of $\partial\Delta$. Then

$$d(F_1, F_2) \leqq d(F_1, G_1) + d(G_1, G_2) + d(G_2, F_2) \leqq 2 + \operatorname{diam}\Delta' = h_1(P) + 1.$$

Thus we have

Corollary 3.4. *If $P = P(d,h)$ then* $\operatorname{diam}\Sigma(P) \leqq h_1(P) + 1$.

The only obstacle to demonstrating that v_1 is in fact a shedding vertex of $\Sigma(P(d,h))$, and hence that $\Sigma(P(d,h))$ is vertex decomposable and satisfies the Hirsch property, is proving that $\operatorname{lk}_{\Sigma(P(d,h))} v_1$ is vertex decomposable. We have, however, not yet succeeded in showing this.

There is no example known of a simplicial polytope P such that $\Sigma(P)$ is not vertex decomposable. This suggests the following question: Does there exist an $h \in h(\mathscr{P}_s^d)$ such that h cannot be the h-vector of *any* vertex decomposable complex? Vertex decomposable simplicial complexes are always shellable and hence are members of a larger class of simplicial complexes called *Cohen–Macaulay complexes* [10]. The h-vector of any Cohen–Macaulay complex is an O-sequence, characterized in the following manner:

For positive integers k and i, k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqq j \geqq 1$. Put

$$k^{\langle i \rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}$$

and define also $0^{\langle i \rangle} = 0$ for positive i. A vector of integers $(h_0, h_1, \ldots, h_d)$ is an *O-sequence* if $h_0 = 1$, $h_i \geqq 0$, $1 \leqq i \leqq d$, and $h_{i+1} \leqq h_i^{\langle i \rangle}$, $1 \leqq i \leqq d-1$.

Thus the h-vector of any vertex decomposable simplicial complex must be an O-sequence. We can, however, use Theorem 3.2 to show the converse:

THEOREM 3.5. *Let $h = (h_0, h_1, \ldots, h_d)$ be any O-sequence. Then there exists a vertex decomposable simplicial $(d-1)$-complex Δ such that $h(\Delta) = h$.*

PROOF. Let $\nu = h_1 + d$. For each $0 \leqq i \leqq d$ choose the first (in the ordering $<$) h_i elements F of $\mathscr{E}_\nu^{1,d}$ such that $\delta(F) = i$. Call this set $\mathscr{B}_i$ and put $\mathscr{B} = \bigcup_{i=0}^d \mathscr{B}_i$. Then it has been shown [6] by methods almost identical to those in §§2,3 and 6 of [1] that $\mathscr{B}$ satisfies (β) and that $h(\Delta(\mathscr{B})) = h$. Now apply Theorem 3.2. □

Therefore the answer to our question is no, because $\Sigma(P)$ is known to be Cohen–Macaulay for every simplicial polytope P and so $h(P)$ is always an O-sequence. This means that we cannot rule out the boundary complex of a simplicial polytope being vertex decomposable on the grounds of its h-vector alone.

4. Bistellar equivalence

Let Δ be a simplicial $(d-1)$-complex and $z \notin V(\Delta)$. Suppose $A, B \subseteq V(\Delta) \cup \{z\}$ such that $A \in \Delta$, $B \notin \Delta$, $\dim A + \dim B = d-1$, and $\mathrm{lk}_\Delta A = \partial \bar{B}$. We then define $\chi_A \Delta$ to be the simplicial complex $\chi_A \Delta = (\Delta \setminus A) \cup (\bar{B} \cdot \partial \bar{A})$, and say that $\Sigma = \chi_A \Delta$ is obtained from Δ by a *bistellar operation.* The name comes from the fact that $\mathrm{st}(a, A)[\Delta] = \mathrm{st}(a, B)[\Sigma]$, where $a \notin V(\Delta) \cup \{z\}$ and $\mathrm{st}(a, A)[\Delta]$ is by definition the simplicial complex $(\Delta\backslash A)\cup \bar{a}\cdot\partial\bar{A}\cdot \mathrm{lk}_\Delta A$, called the *stellar subdivision* of A in Δ. Thus Σ is obtainable from Δ by a stellar subdivision followed by a particular *inverse stellar subdivision.* Observe that $B\in\Sigma$, $A\not\in\Sigma$, and $\mathrm{lk}_\Sigma B=\partial\bar{A}$, and hence $\Delta=\chi_B\Sigma$. If $\dim A=k$ we call χ_A a (*bistellar*) k-*operation.* The inverse of a k-operation is therefore a $(d-k-1)$-operation.

Suppose $\Delta_2=\chi_{A_r}\chi_{A_{r-1}}\cdots\chi_{A_1}\Delta_1$. Then Δ_1 and Δ_2 are said to be *bistellarly equivalent.* For any two simplicial d-polytopes P and Q, $\Sigma(P)$ and $\Sigma(Q)$ are bistellarly equivalent. In particular, if Q is a geometric d-simplex T^d we can write $\Sigma(P)=\chi_{A_r}\cdots\chi_{A_1}\Sigma(T^d)$. Pachner [8] observes the following: In such a sequence of bistellar operations, $\alpha_{d-i-1}-\alpha_i=h_{i+1}(P)-h_i(P)$, where $\alpha_j=\mathrm{card}\{i:\dim A_i=j\}$, the number of j-operations. McMullen's conditions then imply that $(1,\alpha_{d-1}-\alpha_0,\alpha_{d-2}-\alpha_1,\ldots,\alpha_{d-n}-\alpha_{n-1})$ is an O-sequence, where $n=[d/2]$. Thus

$$r=\sum_{i=0}^{d-1}\alpha_i\geqq\sum_{i=0}^{n-1}(\alpha_{d-i-1}-\alpha_i)=h_n(P)-1.$$

So if we define $l(P)$ to be the smallest integer r for which $\Sigma(P)$ can be derived from $\Sigma(T^d)$ by a sequence of r bistellar operations, and $l_*(d,h)=\min\{l(P):P\in\mathcal{P}_s^d(h)\}$, then necessarily $l_*(d,h)\geqq h_n-1$. The aim of this section is to prove that equality holds in this expression, confirming conjecture 3.11 of [8].

Suppose Δ is a shellable simplicial d-ball, i.e., the facets of Δ can be ordered $F_0,F_1,\ldots,F_r$ such that for $1\leqq k\leqq r$,

$$\bar{F}_k\cap\left(\bigcup_{i=0}^{k-1}\right)\bar{F}_i=\bigcup_{j=1}^{s_k}\bar{G}_j^k,$$

where the G_j^k are s_k distinct $(d-1)$-faces of Δ for some $1\leqq s_k\leqq d$. Let Δ_k be the simplicial d-ball $\bigcup_{i=1}^k\bar{F}_i$, $0\leqq k\leqq r$, and $A_k=\bigcap_{j=1}^{s_k}\bar{G}_j^k$, $1\leqq k\leqq r$, a $(d-s_k)$-face of Δ_{k-1}. Then in fact it can be seen that $\partial\Delta_k=\chi_{A_k}(\partial\Delta_{k-1})$, $1\leqq k\leqq r$, and so $\partial\Delta=\chi_{A_r}\cdots\chi_{A_1}\partial A_0$. Also (see, e.g., [7]), $h(\Delta_0)=(1,0,0,\ldots,0)$ and

$$h_i(\Delta_k)=\begin{cases}h_i(\Delta_{k-1}) & \text{if } i\neq s_k,\\ h_i(\Delta_{k-1})+1 & \text{if } i=s_k.\end{cases}$$

Therefore $\partial\Delta$ can be obtained from $\partial\Delta_0$, the boundary of a d-simplex, by a sequence of $h_1(\Delta)+\cdots+h_{d+1}(\Delta)=f_d(\Delta)-1$ bistellar operations, of which exactly $h_i(\Delta)$ are $(d-i)$-operations.

Now consider the simplicial d-polytope $P=P(d,h)$. As noted in the previous section, $\Sigma(P)=\partial\Delta$ for a particular simplicial d-ball Δ. In [1] it is shown that Δ is shellable and $h(\Delta) = (1, h_1 - h_0, h_2 - h_1, \ldots, h_n - h_{n-1}, 0, 0, \ldots, 0)$. Thus $\Sigma(P)$ can be obtained from the boundary of a d-simplex by a sequence of $\Sigma_{i=1}^{d+1} h_i(\Delta) = h_n - 1$ bistellar operations, confirming that $l(P) = h_n - 1$ and hence that

THEOREM 4.1. $l_*(d, h) = h_n - 1$.

We remark that for the particular shelling order of Δ described in [1] we can at each stage in the shelling process insure that $\partial \Delta_k$ is realizable as the boundary complex of some simplicial d-polytope. Hence we have exhibited a *geometric bistellar equivalence* [8] between $\Sigma(P)$ and the boundary of a d-simplex in $h_n - 1$ steps.

Note added in proof. Kleinschmidt has identified a simplicial polytope with non-vertex-decomposable boundary complex.

REFERENCES

1. L. J. Billera and C. W. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Comb. Theory, Ser. A **31** (1981), 237–255.

2. L. J. Billera and C. W. Lee, *The numbers of faces of polytope pairs and unbounded polyhedra*, Eur. J. Comb. **2** (1981), 307–322.

3. G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, 1963.

4. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.

5. V. Klee and D. W. Walkup, *The d-step conjecture for polyhedra of dimension $d < 6$*, Acta Math. **117** (1967), 53–78.

6. C. W. Lee, *Counting the faces of simplicial convex polytopes*, Ph.D. Thesis, Cornell University, Ithaca, New York, 1981.

7. P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, London Math. Soc. Lecture Note Series 3, Cambridge University Press, 1971.

8. U. Pachner, *Über die bistellar Äquivalenz simplizialer Sphären und Polytope*, Math. Z. **176** (1981), 565–576.

9. J. S. Provan and L. J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), 576–594.

10. R. P. Stanley, *Cohen–Macaulay complexes*, in *Higher Combinatorics* (M. Aigner, ed.), D. Reidel, Dordrecht-Holland, 1977, pp. 51–62.

11. R. P. Stanley, *The number of faces of a simplicial convex polytope*, Advances in Math. **35** (1980), 236–238.

IBM THOMAS J. WATSON RESEARCH CENTER
YORKTOWN HEIGHTS, NY 10598 USA
AND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KENTUCKY
LEXINGTON, KY 40506 USA